

On the degree distribution of a growing network model

Linda Farczadi
University of Waterloo
lindafarczadi@gmail.com

Nicholas Wormald*
Monash University
nick.wormald@monash.edu

Abstract

In this note we make some specific observations on the distribution of the degree of a given vertex in certain model of randomly growing networks. The rule for network growth is the following. Starting with an initial graph of minimum degree at least k , new vertices are added one by one. Each new vertex v first chooses a random vertex w to join to, where the probability of choosing w is proportional to its degree. Then k edges are added from v to randomly chosen neighbours of w .

1 Introduction

In this note we make some specific observations on the distribution of the degree of a given vertex in certain model of randomly growing networks. Fix an integer $k \geq 2$. We start with a seed graph G_k^1 consisting of one vertex v_1 with k loops. For $t \geq 2$ given G_k^{t-1} we obtain G_k^t as follows:

- we add a new vertex v_t which connects first to an existing vertex v_i chosen by preferential attachment and then to $k - 1$ neighbours of v_i chosen uniformly at random.

We let V_t and E_t denote the vertex set and edge set of G_k^t . Note that $|V_t| = t$ and $|E_t| = kt$. We denote by $d_t(v_i)$ the degree of vertex v_i in G_k^t . We define $N_t(v_i)$ to be the set of neighbours of vertex v_i in G_k^t .

We begin with a simple derivation of the expected value of $d_t(v_i)$ in Section 2.1, then describe some closely related existing results, and apply them to get a more precise description of the distribution of $d_t(v_i)$ in Section 2.4.

It is straightforward to modify our results for any given initial seed graph.

2 Degree distribution

2.1 Expected degree of a given vertex

Fix vertex v_i . We want to study $d_n(v_i)$, the degree of vertex i at the n^{th} step in the process. For $t \geq i$ we have

$$\begin{aligned} \mathbf{P}(v_{t+1} \text{ connects to } v_i | d_t(v_i)) &= \frac{d_t(v_i)}{2|E_t|} + \sum_{v_j \in N_t(v_i)} \frac{d_t(v_j)}{2|E_t|} \frac{k}{d_t(v_j)} \\ &= \frac{d_t(v_i)}{2kt} + d_t(v_i) \left(\frac{k-1}{2kt} \right) \\ &= \frac{d_t(v_i)}{2t}. \end{aligned}$$

Taking expectations of both sides gives

$$\mathbf{P}(v_{t+1} \text{ connects to } v_i) = \frac{\mathbf{E}[d_t(v_i)]}{2t}.$$

*Supported by an ARC Australian Laureate Fellowship. Research supported partly by NSERC

We then have

$$\begin{aligned}
\mathbf{E}[d_{t+1}(v_i)] &= \mathbf{E}[d_t(v_i)] + \mathbf{P}(v_{t+1} \text{ connects to } v_i) \\
&= \mathbf{E}[d_t(v_i)] + \frac{\mathbf{E}[d_t(v_i)]}{2t} \\
&= \left(1 + \frac{1}{2t}\right) \mathbf{E}[d_t(v_i)].
\end{aligned}$$

Since each vertex has degree k when it joins the graph we have $\mathbf{E}[d_i(v_i)] = k$. We obtain for $1 \leq i \leq n$

$$\begin{aligned}
\mathbf{E}[d_n(v_i)] &= k \prod_{t=i}^{n-1} \left(1 + \frac{1}{2t}\right) \\
&= \frac{k\Gamma(i)\Gamma(n+1/2)}{\Gamma(n)\Gamma(i+n/2)} \\
&= k\sqrt{n/i}(1 + O(1/i)).
\end{aligned}$$

2.2 LCD model of Bollobás and Riordan

The LCD model of Bollobás and Riordan can be described as follow: start with G_1^1 the graph with one vertex and one loop; for $t \geq 2$ given G_1^{t-1} obtain G_1^t by adding one vertex v_t and one edge connecting v_t to an existing vertex v_i chosen randomly with probability given by

$$\mathbf{P}(v_i = s) = \begin{cases} \frac{d_{t-1}(s)}{2t-1} & \text{if } 1 \leq s \leq t-1 \\ \frac{1}{2t-1} & \text{if } s = t \end{cases}$$

Then for a given parameter $k > 1$ obtain G_k^n by first constructing G_1^{kn} on vertices $v'_1, v'_2, \dots, v'_{kn}$ using the process described above. Then identify vertices v'_1, \dots, v'_k to form vertex v_1 of G_k^n , vertices v'_{k+1}, \dots, v'_{2k} to form vertex v_2 , and so on.

We observe that both the k -neighbour model and the Bollobás-Riordan model satisfy the following condition

$$\mathbf{P}(v_{t+1} \text{ connects to } v_i | d_t(v_i)) = \frac{d_t(v_i)}{\sum_{j=1}^t d_t(v_j)}.$$

This is known as the Barabási-Albert (BA) description. Hence the degree of a given vertex has the same distribution in both models. In particular the following result concerning the degree sequence of Bollobás and Riordan [BRS⁺01], [BR03] applies to the k -neighbour model as well.

Theorem 1. *Let $N_n(d)$ be the number of vertices of degree d in G_k^n and define*

$$\alpha(k, d) = \frac{2k(k+1)}{d(d+1)(d+2)}.$$

Then for a fixed $\epsilon > 0$ and $0 \leq d \leq n^{1/15}$ the following holds with high probability

$$(1 - \epsilon)\alpha(k, d) \leq N_n(d) \leq (1 + \epsilon)\alpha(k, d).$$

2.3 General preferential attachment models of Ostroumova et al.

We can obtain results about the k neighbour model by observing that it belongs to a certain class of general preferential attachment models. Specifically Ostroumova et al. [ORS12] define the PA-class by considering all random graph models \mathcal{G}_k^n that fit the following description:

- G_k^n is a graph with n vertices and kn edges obtained from the following random graph process: start at time n_0 with an arbitrary seed graph $G_k^{n_0}$ with n_0 vertices and kn_0 edges; at time t obtain the graph G_k^t from G_k^{t-1} by adding a new vertex and k edges connecting this vertex to some k vertices of G_k^{t-1} .

Then \mathcal{G}_k^n belongs to the class PA-class if it satisfies the following conditions for some constants A and B :

$$\mathbf{P}(d_{t+1}(v_i) = d_t(v_i) | G_k^t) = 1 - A \frac{d_t(v_i)}{n} - B \frac{1}{n} + O\left(\frac{(d_t(v_i))^2}{n^2}\right) \quad (1)$$

$$\mathbf{P}(d_{t+1}(v_i) = d_t(v_i) + 1 | G_k^t) = A \frac{d_t(v_i)}{n} + B \frac{1}{n} + O\left(\frac{(d_t(v_i))^2}{n^2}\right) \quad (2)$$

$$\mathbf{P}(d_{t+1}(v_i) = d_t(v_i) + j | G_k^t) = O\left(\frac{(d_t(v_i))^2}{n^2}\right) \quad 2 \leq j \leq k \quad (3)$$

$$\mathbf{P}(d_{t+1}(v_{t+1}) = k + j | G_k^t) = O\left(\frac{1}{n}\right) \quad 1 \leq j \leq k \quad (4)$$

Then we can observe that our model belongs to this PA-class with parameters $A = 1/2$ and $B = 0$. Then the following two results from [ORS12] apply to our model.

Theorem 2. *Let $N_n(d)$ be the number of vertices of degree d in G_k^n and $\theta(X)$ be an arbitrary function such that $|\theta(X)| < X$. There exists a constant $C > 0$ such that for any $d \geq k$ we have*

$$\mathbf{E}[N_n(d)] = \alpha(k, d) (n + \theta(Cd^4))$$

where

$$\alpha(k, d) = \frac{2k(k+1)}{d(d+1)(d+2)} \sim 2k(k+1)d^{-3}.$$

Theorem 3. *For any $\delta > 0$ there exists a function $\psi(n) = o(n)$ such that for any $k \leq d \leq n^{\frac{1}{8}-\delta}$*

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(|N_n(d) - \mathbf{E}[N_n(d)]| \geq \frac{\psi(n)}{d^3}\right) = 0.$$

2.4 Urn models

We can obtain the distribution of $d_n(v_i)$ by using an urn model. Our urn contains balls of two colours: white and black. White balls represent edge-ends incident with vertex i and black balls represent edge-ends not incident with vertex i . Suppose that the urn initially has a_0 white balls and b_0 black balls where

$$\begin{aligned} a_0 &= k \\ b_0 &= 2i - k \\ t_0 &= a_0 + b_0 = 2i. \end{aligned}$$

At each step, one ball is drawn randomly from the urn. If the drawn ball is white, replace it and put an additional α white and $\sigma - \alpha$ black. If it is black, replace and put σ more black balls.

We now introduce some relevant results about urn models from Flajolet et al. [FDP06]. Consider a triangular urn with replacement matrix

$$\begin{pmatrix} \alpha & \sigma - \alpha \\ 0 & \sigma \end{pmatrix}$$

Let $H_n \binom{a_0 \ a}{b_0 \ b}$ be the number of histories of length n that start in configuration (a_0, b_0) and end in configuration (a, b) . Note that $a + b = a_0 + b_0 + \sigma n$. Then the generating function of urn histories is defined as

$$H(x, 1, z) := \sum_{n, a} H_n \binom{a_0 \ a}{b_0 \ b} x^a \frac{z^n}{n!}$$

and is given by

$$H(x, 1, z) = x^{a_0} (1 - \sigma z)^{-b_0/\sigma} \left(1 - x^\alpha \left(1 - (1 - \sigma z)^{\alpha/\sigma} \right) \right)^{-a_0/\alpha}.$$

Letting $\Delta := (1 - \sigma z)^{-1/\sigma}$ we have from [FDP06, Equation (74)]

$$H(x, 1, z) = x^{a_0} \Delta^{b_0} \left(1 - x^\alpha (1 - \Delta^{-\alpha}) \right)^{-a_0/\alpha}$$

(The reader may notice that [FDP06] is not entirely consistent, but in that paper, the ‘balls of the first type’ do always correspond to the first row of the replacement matrix.) Now let A_m be the number of white balls in the urn after m trials. The probability that this equals $a_0 + x\alpha$ for some $0 \leq x \leq m$ is given by [FDP06, Equation (75)] as

$$\mathbf{P}(A_m = a_0 + x\alpha) = \binom{x + \frac{a_0}{\alpha} - 1}{x} \sum_{i=0}^x (-1)^i \binom{x}{i} \frac{[m + (b_0 - \alpha i)/\sigma - 1]_m}{[m + t_0/\sigma - 1]_m}$$

where $[\cdot]_m$ denotes falling factorial.

We next explain why the urn results apply to the random network process. At any given step, let s denote the total degrees of the vertices. If a vertex v has degree d then the probability it is chosen as the first vertex is d/s . The probability it is chosen as the second vertex via any given one of its d incident edges is $(k-1)/s$. Hence, the probability it receives a new edge is dk/s . (This is similar to the derivation in Section 2.1.) So we may use an urn with kd white balls and s black balls. At each step, s increases by $2k$, whilst kd increases by k if v receives a new edge, and by 0 otherwise. Hence, at each step the number of white balls is k times the degree of v , and the parameters are $\alpha = k$ and $\sigma = 2k$. The initial number of white balls, a_0 , is k times the initial degree of the vertex, and the initial number of black balls is $2k$ times the initial number of edges.

Suppose the initial graph is a copy of K_j . Then for any of the j initial vertices we have $\alpha = k$, $a_0 = k(j-1)$, $t_0 = j(j-1)$ and $b_0 = t_0 - a_0 = (j-k)(j-1)$. Also $\sigma = 2k$. For each of those j vertices v_i (which initially have degree $j-1$), there are $m = n - j$ trials. So the probability that the corresponding urn process finishes with $k(j-1+x)$ balls is

$$\mathbf{P}(d_n(v_i) = j-1+x) = \binom{x+j-2}{x} \sum_{u=0}^x (-1)^u \binom{x}{u} \frac{[n-j + \frac{(j-1)^2 - ku}{2k} - 1]_{n-j}}{[n-j + R - 1]_{n-j}}$$

where $R = j(j-1)/(2k)$. On the other hand, if $i > j$ then $t_0 = j(j-1) + 2(i-j)k$, $a_0 = k^2$, $b_0 = t_0 - a_0$, $m = n - i$ and so

$$\mathbf{P}(d_n(v_i) = k+x) = \binom{x+k-1}{x} \sum_{u=0}^x (-1)^u \binom{x}{u} \frac{[n-j + R - (u+k)/2 - 1]_{n-i}}{[n-j + R - 1]_{n-i}}.$$

We can also obtain the moments of A_n directly from the generating function of urn histories. In particular, as explained in [FDP06], the first and second moments are given by

$$\begin{aligned} \mathbf{E}[A_n] &= \frac{\Gamma(n+1)\Gamma(\frac{t_0}{\sigma})}{\sigma^n \Gamma(\frac{t_0}{\sigma} + n)} [z^n] \left(\frac{\partial H(x, 1, z)}{\partial x} \right)_{x=1} \\ &= \frac{\Gamma(n+1)\Gamma(\frac{t_0}{\sigma})}{\sigma^n \Gamma(\frac{t_0}{\sigma} + n)} [z^n] a_0 \Delta^{t_0+\alpha} \\ \mathbf{E}[A_n^2] &= \frac{\Gamma(n+1)\Gamma(\frac{t_0}{\sigma})}{\sigma^n \Gamma(\frac{t_0}{\sigma} + n)} [z^n] \left(\frac{\partial^2 H(x, 1, z)}{\partial x^2} \right)_{x=1} \\ &= \frac{\Gamma(n+1)\Gamma(\frac{t_0}{\sigma})}{\sigma^n \Gamma(\frac{t_0}{\sigma} + n)} [z^n] (a_0(a_0 + \alpha) \Delta^{t_0+2\alpha} - a_0(\alpha+1) \Delta^{t_0+\alpha}). \end{aligned}$$

Performing coefficient extraction gives

$$\begin{aligned}
[z^n] a_0 \Delta^{t_0+\alpha} &= a_0 [z^n] (1 - \sigma z)^{-\frac{t_0+\alpha}{\sigma}} \\
&= a_0 \binom{-\frac{t_0+\alpha}{\sigma}}{n} (-\sigma)^n \quad \text{using } [t^n] (1 + at)^r = \binom{r}{n} a^n \\
&= a_0 \binom{n + \frac{t_0+\alpha}{\sigma} - 1}{n} \sigma^n \quad \text{using } \binom{-r}{n} = \binom{n+r-1}{n} (-1)^n \\
&= a_0 \sigma^n \frac{\Gamma(n + \frac{t_0+\alpha}{\sigma})}{\Gamma(n+1) \Gamma(\frac{t_0+\alpha}{\sigma})} \quad \text{using } \binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1) \Gamma(x-y+1)}
\end{aligned}$$

And by linearity:

$$\begin{aligned}
[z^n] (a_0 (a_0 + \alpha) \Delta^{t_0+2\alpha} - a_0 (\alpha + 1) \Delta^{t_0+\alpha}) &= a_0 (a_0 + \alpha) [z^n] \Delta^{t_0+2\alpha} - a_0 (\alpha + 1) [z^n] \Delta^{t_0+\alpha} \\
&= a_0 \left[(a_0 + \alpha) \sigma^n \frac{\Gamma(n + \frac{t_0+2\alpha}{\sigma})}{\Gamma(n+1) \Gamma(\frac{t_0+2\alpha}{\sigma})} - (\alpha + 1) \sigma^n \frac{\Gamma(n + \frac{t_0+\alpha}{\sigma})}{\Gamma(n+1) \Gamma(\frac{t_0+\alpha}{\sigma})} \right].
\end{aligned}$$

Plugging these coefficients back in the equations for the first and second moment we obtain

$$\begin{aligned}
\mathbf{E}[A_n] &= \frac{a_0 \Gamma(\frac{t_0}{\sigma}) \Gamma(n + \frac{t_0+\alpha}{\sigma})}{\Gamma(\frac{t_0+\alpha}{\sigma}) \Gamma(\frac{t_0}{\sigma} + n)} \\
\mathbf{E}[A_n^2] &= a_0 \left[\frac{(a_0 + \alpha) \Gamma(\frac{t_0}{\sigma}) \Gamma(n + \frac{t_0+2\alpha}{\sigma})}{\Gamma(\frac{t_0+2\alpha}{\sigma}) \Gamma(\frac{t_0}{\sigma} + n)} + \frac{(\alpha + 1) \Gamma(\frac{t_0}{\sigma}) \Gamma(n + \frac{t_0+\alpha}{\sigma})}{\Gamma(\frac{t_0+\alpha}{\sigma}) \Gamma(\frac{t_0}{\sigma} + n)} \right]
\end{aligned}$$

Applying these results to our example where $a_0 = k$, $b_0 = 2i - k$, $\alpha = 1$, $\sigma = 2$ and the number of trials is $n - i$ we obtain

$$\begin{aligned}
\mathbf{E}[d_n(i)] &= \frac{k \Gamma(i) \Gamma(n + 1/2)}{\Gamma(i + 1/2) \Gamma(n)} \\
\mathbf{E}[d_n(i)^2] &= k \left[\frac{(k+1) \Gamma(i) \Gamma(n+1)}{\Gamma(i+1) \Gamma(n)} + \frac{2 \Gamma(i) \Gamma(n+1/2)}{\Gamma(i+1/2) \Gamma(n)} \right] \\
&= \frac{k(k+1)n}{i} + 2 \mathbf{E}[d_n(i)].
\end{aligned}$$

We note that the value for the first moment matches the one obtained previously by solving the simple recursion in Section 2.1.

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